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NUMERICAL METHOD/ANALYSIS STUDENTS' CONCEPTUAL DERIVATIVE KNOWLEDGE

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ABSTRACT

Advanced level mathematics and engineering courses such as Numerical Methods and Numerical Analysis require advanced knowledge of derivative. In this paper, undergraduate and graduate engineering and mathematics Numerical Methods-Analysis course students' conceptual derivative knowledge is observed. Students' ability to determine the cusp points (if exist) of a quotient function after calculating the first and second derivatives of the function is investigated. In addition, student responses to find the derivative of a composition function is evaluated by using the schema development idea of Piaget and Garcia (1989) where the participants were expected to apply the chain rule and determine the domain of differentiation for the given function.

Key Words: Derivative of functions, the chain rule, schema classification, concept image, critical points.

INTRODUCTION

Function derivative is an important fundamental calculus concept that undergraduate and graduate mathematics and engineering students are expected to know to learn more advanced level concepts. In particular, first and second derivative concepts have critical importance in determining the cusp points of a given function and determining the components of the power series of functions. For the past three decades researchers in educational studies worked with students to understand their conceptual derivative knowledge. In this study, a brief review of the derivative literature will be covered. In addition, keeping in mind that the derivative of a function is another function and the graph of a function can be directly related to the derivative knowledge of a student, a review of the function education literature in Research on Undergraduate Mathematics Education (RUME) will be covered. Several research results indicated the benefits of applying the Action-Process-Object-Schema (APOS) theory introduced by Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas (1996) for observing students' levels of understanding the function concept where the derivative concept also plays an important role. However, in several educational studies APOS theory was not found useful to evaluate students' mathematical conceptual understanding. We briefly cover APOS theory and the relevant literature in RUME as a part of this study.

The main goal of this study is to observe derivative knowledge of the undergraduate and graduate mathematics and engineering students who were enrolled either in a Numerical Methods or Numerical Analysis course in a large Midwestern University in the United States of America. Topics to be evaluated include finding cusp points (if exist) after calculating derivatives and applying the chain rule to a composition function that is not necessarily differentiable on the entire real line. The collected data for the derivative calculation related question indicated application of APOS theory is not appropriate to evaluate students' conceptual derivative knowledge for this question. Several examples of participant responses to determine whether the cusp point exist or not will be presented in this work. In addition, the schema development idea of Piaget and Garcia (1989) will be used to evaluate the collected data for the chain rule question.

METHOD

Participants and the Research Problems

In this study 17 senior undergraduate and graduate students' majoring in Engineering and Mathematics disciplines who either completed or enrolled to a Numerical Methods or Numerical Analysis course at a large Midwestern university were asked to complete the same questionnaire in 80 minutes that consisted of 15 questions. Participants were interviewed for approximately 40 minutes and the length of these interviews varied based on the responses of the students' to the questionnaire questions. These questions are designed to cover the concepts such as functions, derivatives of functions, integrals of functions and power series. In addition, several questions were included in the questionnaire to understand participants programming preferences. The concepts considered in this questionnaire were covered in the pre-requisite courses of the Numerical Methods/Analysis courses; therefore, the participants completed the pre-requisite courses with a grade of C (70/100) or higher. The interviews were conducted uniformly across the participants based on their responses. The research problems of this study are designed to cover the following concepts:

- Finding the cusp points of a given function after calculating the derivatives of the given function.
- Finding the derivative of a composition function by applying the chain rule.
  In particular, participants were asked to respond to the following two questions:
- Is h(x)=sin(|x|) differentiable for all real numbers x ? If yes, please explain the domain of it where it can be a differentiable function. If it is not differentiable, please explain why.
- Find the cusp points of f(x)=x/(x+1) after calculating the related derivatives if the cusp points exist.

The evaluation of the collected data will be represented in this paper with examples from student responses. Some of the results obtained for other questions and the evaluation of the corresponding collected data is planned to be published elsewhere.

Schema Development and Derivative Literature Review

Students' conceptual derivative knowledge is a focus point of several researchers (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Ferrini-Mundy & Gaudard, 1992; Orton, 1983; Zandieh, 2000). Orton (1983) (pp. 242) indicated students' being taught differentiation only as a rule without having much conceptual experience with its applications. Basic derivative knowledge is necessary to understand conceptually more developed topics under derivative concept such as the chain rule.

Clark, Cordero, Cottrill, Czarnocha, DeVries, St. John, Tolias, and Vidakovic (1997) used the stages of the triad (intra, inter, and trans stages) to investigate how first year calculus students' construct the chain rule concept. Their attempt to use the APOS theory resulted in insufficiency by itself; therefore, they included the schema development idea of Piaget and Garcia (1989). Clark et al. (1997) had their participating students' chain rule triad classification as follows:

- **Intra Stage:** Students' were able to apply the derivative rules and the chain rule, however they did not know the relationship between them.
- **Inter Stage:** Student's were able apply all different derivative cases and recognize that they are related.
- **Trans Stage:** Students' were able to construct and apply the chain rule.

Cottrill (1999) observed first year calculus students' chain rule knowledge and concluded that most of the participating students' provide an example of what the chain rule is rather than explaining how it works. Wangberg, Engelke, and Karakok (2011) investigated the relationship between students' understanding of composition of functions and the chain rule by studying how students use and interpret the chain rule while working in an online homework environment in their preliminary report. Our goal in this study is to understand engineering and mathematics undergraduate and graduate students’ conceptual derivative knowledge by observing the following two sub-concepts:

- Student success in applying quotient or product rule to a given quotient function,
- Applying the chain rule on a function that is not differentiable.
  In this study, the data collected for the chain rule question will be evaluated by using the following triad classification:
- **Intra Stage:** Students' were classified in this category if they did not know how to apply the chain rule and not able to determine the domain of differentiation.
- **Inter Stage:** Student's were able to apply the chain rule correctly on an interval but not in the entire domain.
- **Trans Stage:** Students' were able to determine the domain of differentiation and applied the chain rule correctly.

APOS Theory

The philosophy of mathematics showed its effect on mathematics education in the undergraduate curriculum in the 1990`s. Piaget`s schemes idea in the 1970`s and its development with detailed explanations by Piaget and Garcia in the 1980`s had an influence on researchers of undergraduate mathematics education curriculum in the 1990`s. Students` conceptual view of the function was defined by Breidenbach, Dubinsky, Hawks and Nichols in 1992 who relied on Piaget`s study of functions in 1977 (Piaget, Grize, Szeminska & Bang, 1977). This formed the action-process-object idea in mathematics education for the undergraduate curriculum. In 1996, Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas applied action, process, object and schema theory (called APOS theory) to mathematical topics (mostly functions) and they explained this theory as the combined knowledge of a student in a specific subject based on Piaget`s philosophy. In this theory, every concept can be constructed on different concepts and schemas. For example, if a researcher would like to work on integration of functions, the researcher can base the schemas on functions, limits of functions, the understanding of the derivation of functions, continuity of functions, and number knowledge of students. All Schema combinations can form a schema. We can also say that every concept requires concept knowledge and the construction of a specific concept depends on knowledge of the other concepts. Some of the researchers such as Clark, Cordero, Cottrill, Czarnocha, DeVries, St. John, Tolias & Vidakovic (1997) didn't find the APOS theory applicable in analyzing data in their research. Baker, Cooley and Trigueros (2000) based their research on APOS theory when they focused on undergraduate students conceptual function knowledge with a calculus graphing problem. In 2007, Cooley, Trigueros and Baker built on their work from 2000 (Baker et al. (2000)) by focusing on the thematization of the schema with the intent to expose those possible structures acquired at the most sophisticated stages of schema development. For a detailed review of the APOS theory see Dubinsky and McDonald (2002).

In the last decade APOS theory is widely used in several educational research areas. It is used by Parraguez and Oktac (2010) to lead the students' towards constructing the vector space concept, Mathews and Clark (2007) to observe successful students' conceptual knowledge of mean, standard deviation, and the central limit theorem who completed an elementary statistics course with a grade of "A", Kashefi, Ismail, and Yusof (2010) to observe students' obstacles in the learning of two variable functions in calculus.

RESULTS

In this section, the collected data will be evaluated and presented in two sections: A section of participating student responses to find the cusp points (if exist) of a quotient function and their calculations, and a section of participating student responses to a chain rule question with the collected data evaluated by using triad classification.

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Derivative Calculations and Critical Points

In this study, participants were asked to find the local maximum, local minimum and inflection points of f(x)=x/(x+1) if they exist. Given f(x) as a quotient function, participants were expected to find the first and second derivatives of the given function by using the product or quotient rule as a part of their responses. During the interviews, students were asked to calculate the first and second derivatives of f(x) if they did not calculate it prior to the interviews. Prior to the interviews, RP 4 and RP 9 miscalculated the first derivative of f(x), and RP 4 found critical points of f(x) but could not explain the reason during the interview:

I: Okay, so we have this information. And what is the meaning of local maximum, local minimum and inflection point, do you remember?

RP 4: I think the local maximum and local minimum is the first derivative. I don't remember.

I: It's related to first derivative.

RP 4: I think so.

I: Okay, what do we do with the first derivative, do you remember?

RP 4: No.

I: Okay, if I give you the definition, will you be able to find?

RP 4: I think so, yes.

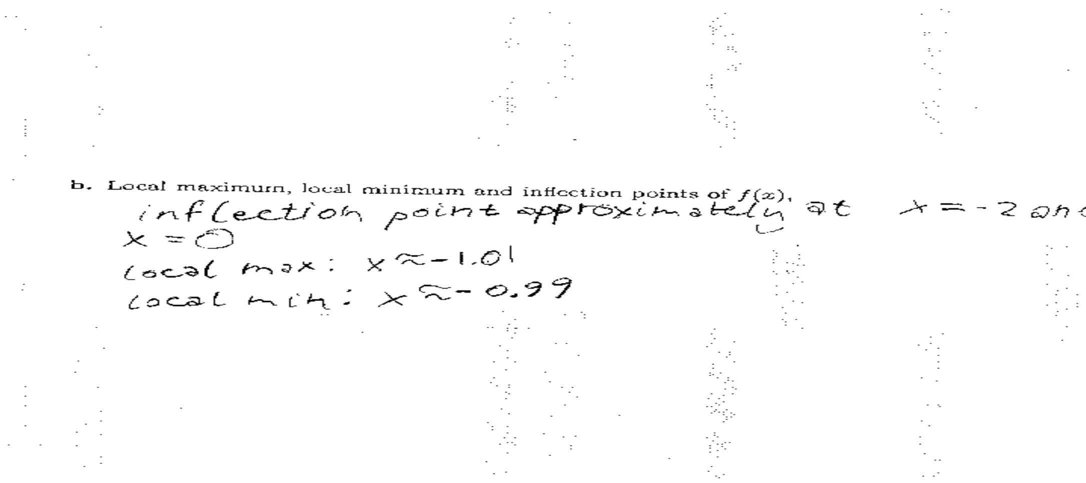
I: When first derivative of the given function is equal to zero, for those x values, we have either a local maximum, local minimum or inflection point.

RP 4: Oh, okay.

I: So, in this case what would be the first derivative?

RP 4: (Writes

$$f'(x) = \frac{1(x+1)}{(x+1)^2} - x^2$$



b. Local maximum, local minimum and inflection points of $f(x)$.

inflection point approximately at $x = -2$ and $x = 0$

local max: $x \approx -1.01$

local min: $x \approx -0.99$

Fig 1: Response of RP 4 to the critical point question.

Prior to the interview, two of the participants found the local maximum and local minimum of f(x) to be infinity and negative infinity, respectively:

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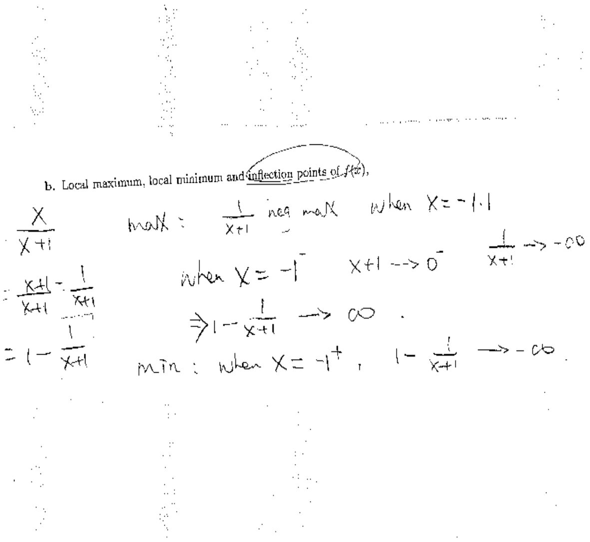

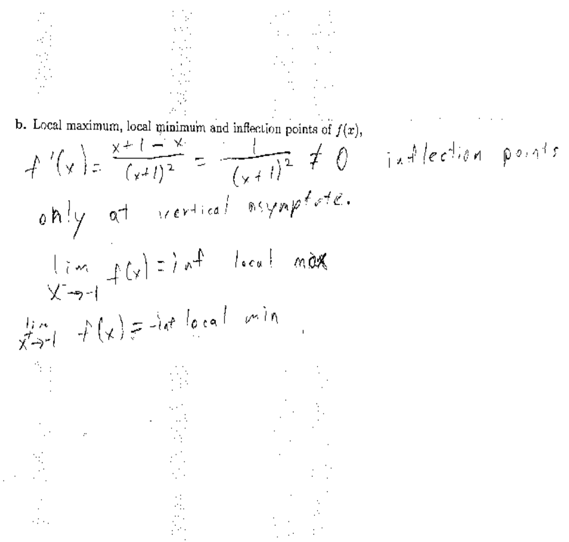

Fig 2: Response of RP 5 to critical point question

Fig 3: Response of RP 11 to the critical point question.

24% (4/17) of the participants miscalculated the second derivative of f(x).

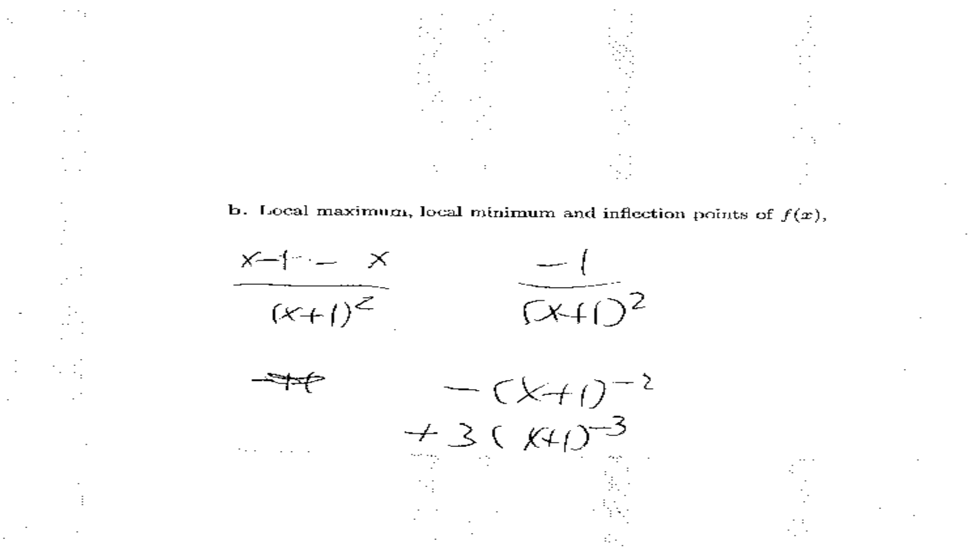

Fig 4. Derivative calculation of RP 7

The Chain Rule

In this study, conceptual chain rule knowledge of the participants was determined by evaluating the participant responses to the following non-routine chain rule question:

Question 2: Is h(x)=sin(|x|) differentiable for all real numbers x ? If yes, please explain the domain of it where it can be a differentiable function. If it is not differentiable, please explain why.

59% (10/17) of the participants answered the chain rule question correct, forming the trans level schema classification; however, one of these participants did not know the definition of |x| even though his/her response was correct.

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I: And let me ask you, here you are answering sin(|x|) differentiable or not. And you are saying "If x>0, |x| is x and if x<0, |x|=x."
RP 12: Absolute value of x is still x.
I: Okay. And it's not differentiable because of?
RP 12: The, especially the point at zero.
I: I see.
RP 12: Because if we expand the graph at the point you'll have a sharp point...

One of the participants declared why sin(|x|) is not differentiable by stating "No, can't take derivative of absolute values"; however, he/she could not justify his/her response during the interview.

I: ... so here we have sin(|x|) differentiable for all real numbers and you are saying "No, can't take derivative of absolute values." What is the idea behind it, can you explain me a little bit?
RP 4: I don't know.
I: ... Okay, so you are saying absolute value cannot be differentiable.
RP 4: I guess.
I: And if that is the case do you remember where it is not differentiable, where we cannot find the derivative?
RP 4: No.
I: Do you remember the graph of |x|?
RP 4: Yes.
I: Can you draw it for me?
RP 4: (Sketches the graph of |x|)
I: Okay, so here do you see anywhere that it may not be differentiable? where we cannot find the derivative?
RP 4: (Points the origin on the graph of |x| above)
I: Can you explain me briefly why we cannot take derivative?
RP 4: It's not continuous, it has got like a, an angle.
...
29% (5/17) of the participants did not have the right answer to the chain rule question. 60% (3/5) of these participants did not have the right response to this question prior to the interviews as a result of the image misconception of sin(|x|) in their minds.

I: So here you say "No, at every π the graph is not smooth." What does that mean?
RP 3: So, generally from like a calculus stand point of view it is differentiable if it is smooth. So if we take limit as x approaches some value from left that is going to be equal to the limit as x approaches the same value from the right. So with this we have (Points Figure 5) this. So it is not differentiable at these points (pointing the corner points (pointing 0, π, 2π).)

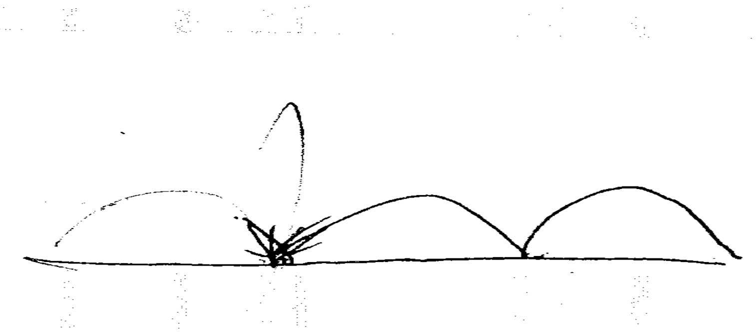

Fig 5: Graph of RP 3 for h(x)=sin(|x|).

I: And what is the reason for that? Is there any specific name for it? When it not differentiable like this? Do you recall?

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RP 3: Point wise non-differentiable, something like that.
I: If I say you corner point. Does it...
RP 3: Yeah, I've heard that. It is kind of a, I mean yeah. It is kind of a Calc 1 kind of explanation for what that is, so.
I: So can this function be differentiable at all.
RP 3: Well, yeah, I mean, it will be differentiable if you just went from that (writes [0,π].) Because it's differentiable at every point except for these points. So if you gave a domain that did not include these points, in a sense or if you gave a domain that didn't count these points from both sides, you can go from here to here. You know.
...

I: Is this (pointing sin(|x|)) a differentiable function? You said it is not a differentiable function. So what makes it non-differentiable?
RP 5: I think these points (pointing the corner points on the x-axis of

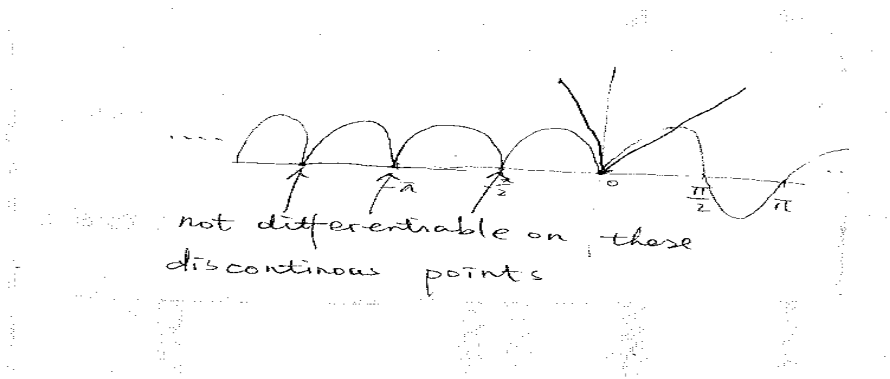

Fig 6: Response of RP 5 to the chain rule question.

) It is like this shaped which makes it non-differentiable (drawing two 2 dimensional cones and pointing corner points of them)
I: Okay. And those are the corner points, at this point. Is |x| a differentiable function? Do you remember?
RP 5: Do you mean this one? No, at this point it is not differentiable. (pointing the corner point at π)
I: Right, okay...

One of the participants who did not have the right graph of h(x) in his written response corrected the graph during the interview:
I: Alright, here the question says is sin(|x|) differentiable? And you are saying "No." What is your reason?
RP 15: I saw you fill through this and I realized there is a diamond here. So it would be differentiable.
I: For what values could it be differentiable if it is differentiable?
RP 15: All values.
I: All values? Let me ask you this: Is sine a differentiable function?
RP 15: Yes.
I: Is absolute value of x a differentiable function?
RP 15: Yes. Oh, no it is not.
I: Where it is not differentiable?
RP 15: At zero. (Draws the graph of y=|x|), okay.
I: So in this case, is it differentiable?
RP 15: Yes, I thought. I thought at zero it was okay. (Pointing the graph drawn in figure 7)
I: So is it differentiable for all x values?

RP 15: Yes... It's not. At x=0.
I: It's not at x=0?
RP 15: Right.
I: Okay. So is there anything else you would want to add to that?
RP 15: No. I think it is fine now.
...

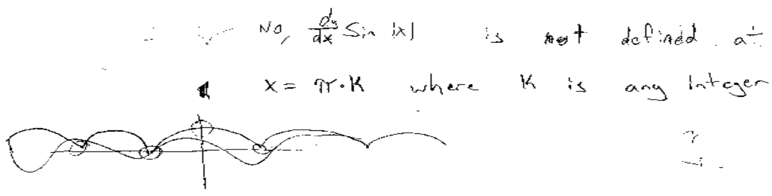

Fig 7: Response of RP 15 to the chain rule question.

CONCLUSION

In this study, undergraduate and graduate engineering and mathematics Numerical Methods and Numerical Analysis courses students' conceptual derivative knowledge is observed by determining their ability to calculate the derivative of a given function to find the cusp points of a quotient function, and their ability to apply the chain rule on a nontraditional calculus problem. The chain rule question in this study is evaluated by using the schema classification.

Vinner (1992) and Aspinwall et al. (1997) observed misconception of students' who encountered difficulty in changing their concept image (in their minds) where these wrong images conflicted with the given information. In this study, image misconception of h(x)=sin(|x|) resulted in misconception of the chain rule similar to the findings of Vinner (1992) and Aspinwall et al. (1997). Both Slavit (1995) and Baker et al. (2000) observed college students' difficulty in functions with cusp points or functions that are not polynomials. We observed a similar difficulty of some of the participants. Several participants had the wrong concept image of h(x)=sin(|x|) in their minds which resulted in derivative misconception. Some of the participants faced difficulty in algebraic calculations of the quotient function f(x)=x/(x+1) to find its cusp points.

Using the triad classification explained previously similar to that of Clark et al. (1997), post interview results indicated 71% (12/17) of the participants fell into the Trans Stage, 18% (3/17) of the participants fell into the inter stage, and 12% (2/17) of the participants fell into the intra stage. On the contrary to the findings of Asiala et al. (1996) we observed some of the students' having difficulty in displaying a relationship between the slope of the tangent line and the first derivative based on the given derivative information and the graphs they sketched. As a result of this study, while calculating the derivatives of the given quotient function and applying the chain rule to h(x)=sin(|x|), the most difficulty is encountered by the senior undergraduate and graduate engineering students' among the participants.

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